Fixed-income average options: a pricing approach based on mean-reverting seasonal models

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Abstract

This paper values fixed-income (discrete- and continuous-time) European Asian and Australian options. We assume that the term structure of interest rates (TSIR) is modeled by the specification proposed in Moreno, Novales, and Platania (2018). We obtain closed-form expressions for the premiums of geometric average options and, for arithmetic average options, premiums are computed by numerical methods.

Keywords: Average options, mean-reversion, Fourier series.

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1 Introduction

This paper prices average call options on zero-coupon bonds and fixed strike assuming that the term structure of interest rates (TSIR) is given by the model proposed in Moreno, Novales, and Platania (2018) (Moreno *et al.* (2018), from now on).[1]

In more detail, we value Asian and Australian options. The pay-off of Asian options depends on the average of an underlying asset price over some predetermined time interval and the pay-off of Australian options depends on the ratio of this average asset price and the underlying asset price at option maturity. We assume that the average price can be geometric or arithmetic, either in discrete- or continuous-time.

These options constitute an important family of derivatives with many applications. Moreover, there are some advantages in trading Asian options as, for instance, a) their pay-off is less volatile as it depends on an average price instead of a terminal one, implying that Asian options are cheaper than (equivalent) European options and b) it is more difficult to manipulate an average price than a final one. As a result, these options can be an adequate hedging instrument for traders who act continuously over finite periods.

Since it can be difficult (or even impossible) to find the probability distribution of the average price of an underlying asset, obtaining an analytical pricing expression for these options can be a challenging task. The general conclusion of the academic literature is that geometric Asian options can be valued analytically under certain assumptions while pricing arithmetic Asian options requires numerical methods as, for instance, Monte Carlo simulations, numerical integration, or approximations of the true distribution of the average price.

Some examples of Asian stock options pricing departing from Black-Scholes assumptions can be found in Kemna and Vorst (1990), Turnbull and Wakeman (1991), Ritchken *et al.* (1993), Geman and Yor (1993), Rogers and Shi (1995), Boyle *et al.* (1997), Angus (1999), Fusai (2004), Linetsky (2004), Sun *et al.* (2013), Cai, Song, and Kou (2015), and Cui *et al.* (2018). As Black-Scholes assumptions are typically not empirically satisfied (specially the constant volatility hypothesis), stochastic

[1] We just price call options because, after this pricing, a similar analysis for put options can be obtained in a straightforward way by applying the put-call parity for these options.

volatility models have been proposed to value Asian options. Some examples are Wong and Cheung (2004), Hubalek and Sgarra (2011), Kim and Wee (2014), Shi and Yang (2014), and Ewald *et al.* (2020).

Moreno and Navas (2008) introduced Australian options, that generalize Variable Purchase Options (VPOs), presented and priced in Handley (2000). These options give the right to buy a stochastic number of shares that depends on the shares price at option's maturity. Standard VPOs are designed to finish "in-the-money" but caps and floors on the number of shares can be included.

The underlying asset of Australian options is a ratio that can be, alternatively, a) the underlying asset price at option's maturity divided by the average asset price during the option's life or b) the inverse of the previous ratio. Assuming that the asset price follows a geometric Brownian motions, Moreno and Navas (2008) provided analytical "Black-Scholes type" expressions for the geometric case while, in the arithmetic case, they applied different numerical pricing techniques.

The continuous-time models that have been proposed to analyze the TSIR can be classified in endogenous and exogenous. Their main features are the following:

- Endogenous models assume that one or more state variables drive the TSIR and, typically, they provide a reasonable analytical tractability and an easy numerical implementation. However, they do not take into account the information embedded in observed interest rates, they do not achieve a good fit to the observed term structure, and their determination of the market risk premium can lead to some arbitrage opportunities. Examples of one-factor models are those proposed in Vasicek (1977) and Cox *et al.* (1985).

- Exogenous models take the TSIR as given and try to achieve a perfect fit to observed interest rates. In contrast to the previous models, they do not need to make any assumption on the market risk premium but, in general, it is difficult to obtain an analytical derivatives pricing and their practical implementation is computationally demanding as, in many cases, they deal with non-Markovian processes.

In this paper we use the one-factor model for the instantaneous interest rate presented in Moreno *et al.* (2018) that generalizes the model proposed in Vasicek (1977).

These authors assumed that the instantaneous risk-free interest rate converges to a certain long-term value (mean-reversion level) that is given by a Fourier series and, then, this value displays a cyclical behavior. Under this functional form, they checked that interest rates follow a Gaussian distribution and, then, the price of a zero-coupon bond follows a lognormal distribution. Also, they provided a closed-form pricing expression for any fixed income derivative and showed that a great flexibility in interest rates is achieved, allowing for a better empirical behavior while maintaining its analytical tractability.

Under this model, we will provide analytical expressions for the premium of geometric Asian calls and, for arithmetic averages, we will apply numerical methods. We will compare these results with those obtained for the particular case that was mentioned, Vasicek (1977). Later, we will extend the pricing part to Australian call options. Similarly to Asian options, we will obtain closed-form expressions for the premiums of geometric Australian options and we will compute numerically these premiums for the arithmetic case.

This paper is organized as follows. Section 2 presents some general statistical results about the multivariate Gaussian distribution and describes the main features of the derivatives that we will value and some relationships between them. Section 3 introduces the TSIR model proposed in Moreno *et al.* (2018) and some key pricing results that will be employed later. Sections 4 and 5 provide pricing results for, respectively, discrete- and continuous-time (geometric and arithmetic) Asian and Australian options for both the Moreno *et al.* (2018) and Vasicek (1977) models and discuss the obtained results. Finally, Section 6 summarizes our main conclusions and proposes some lines of further research.

2 Preliminary results

In this section we present preliminary results, both statistical and financial, which will be used in the following sections. The statistical results focus on the main properties of the multivariate Gaussian distribution, while the financial ones introduce the derivatives we will value and some relationships between them.

2.1 Multivariate Gaussian distributions

We present some basic definitions and statistical results about the multivariate Gaussian distribution that we will use along this paper. First we introduce the equivalent of the univariate standard Gaussian random variable.

Definition 2.1. *A real random vector* $Z = (z_1, z_2, \dots, z_k)'$ *is called a standard Gaussian random vector if all its components* z_i, $i = 1, \dots, k$ *are i.i.d. and each one follows a standard (zero-mean unit-variance) Gaussian random variable.*

A definition of the multivariate Gaussian that does not involve the distribution function follows.

Definition 2.2. *A real random vector* $Y = (y_1, y_2, \dots, y_k)'$ *is called a Gaussian random vector if there exists a standard Gaussian random vector* $Z \in \mathbb{R}^{l \times 1}$*, a vector* $\mu \in \mathbb{R}^{k \times 1}$*, and* $A \in \mathbb{R}^{k \times l}$*, such that* $Y = AZ + \mu$*. Here,* $\Omega = AA'$ *is the variance-covariance matrix.*

In the case that Ω is singular, the corresponding distribution has no density.

We should remark that the sum of Gaussian random variables does not always follows a Gaussian distribution. The following result presents a characterization in terms of the multivariate Gaussian random variable.

Proposition 2.3. *Let* $\{y_i\}_{i=1,2,\dots,k}$ *be a collection of real-valued random variables. Then,* $Y = (y_1, y_2, \dots, y_k)'$ *follows a multivariate Gaussian distribution if and only if, for all* $b \in \mathbb{R}^{k \times 1}$*, the real random variable* $b'Y$ *follows a univariate Gaussian distribution.*

In this case, let μ *and* Ω *denote, respectively, the mean and variance-covariance matrix of* Y*. Then, the mean and variance of* $b'Y$ *are* $b'\mu$ *and* $b'\Omega b$*, respectively.* ■

2.2 Asian options

We introduce now Asian options either in discrete- or continuous-time, assuming both geometric and arithmetic average and we also present the put-call parity for these options.

Given the time interval $[t, T]$, consider the partition $\{T_0 = t, T_1, \ldots, T_n = T\}$. For $i = 1, 2, \ldots, n$, let r_{T_i} denote the interest rate at time T_i, let $P_i = P(r_{T_i}, T_i, T_b)$ denote the price at time T_i of a zero-coupon bond that matures at time T_b, and let w_i be deterministic functions such that $\sum_{i=1}^n w_i = 1$. Each weight w_i will indicate the importance of the bond price at time T_i in the average price. The simplest case is when $w_i = 1/n, \forall i$.

The weighted geometric and arithmetic average bond prices in discrete-time are given by[2]

$$G^{(n)} = \prod_{i=1}^{n} P_i^{w_i}, \quad A^{(n)} = \sum_{i=1}^{n} w_i P_i \tag{1}$$

In a similar way, we can define weighted (geometric and arithmetic) average bond prices in continuous time. To this aim, let $f(s)$ be a deterministic function defined on the time interval $[t, T]$ such that $\int_t^T f(s)ds = 1$. Similarly to the discrete weights w_i, this function indicates the importance of the bond price at time s in the average price. The simplest case is when $f(s) = 1/(T - t)$, $\forall s \in [t, T]$.

Then, the weighted geometric and arithmetic average bond prices in continuous-time are defined as

$$G^{(\infty)} = \exp\left(\int_t^T f(s) \ln(P_s)\, ds\right), \quad A^{(\infty)} = \int_t^T f(s) P_s\, ds \tag{2}$$

Consider Asian options with strike X, maturity T, and whose underlying is $Aver$, the average price of the aforementioned zero-coupon bond, see equations (1)-(2). For a call (resp., put), the final pay-off is the positive part of $Aver - X$ (resp., $X - Aver$).

Let $c_A(r_t, t, T; T_b)$ and $p_A(r_t, t, T; T_b)$ denote, respectively, the premiums at time t of both options. The following Proposition provides the corresponding put-call parity.

Proposition 2.4. *Under no-arbitrage opportunities, the relation between the premiums of these Asian options is given by*

$$p_A(r_t, t, T; T_b) + P(r_t, t, T)\widetilde{\mathbb{E}}_t[Aver] = c_A(r_t, t, T; T_b) + P(r_t, t, T)X$$

or, equivalently, $c_A(r_t, t, T; T_b) - p_A(r_t, t, T; T_b) = P(r_t, t, T)\left(\widetilde{\mathbb{E}}_t[Aver] - X\right)$, *where* $\widetilde{\mathbb{E}}_t[Aver]$ *denotes the conditional expectation (under the risk-neutral measure* $\tilde{P}$*) at time* t *of the average bond price.* ■

[2] Along this paper we will use the term average to refer to both weighted or unweighted average.

We will see that this conditional expectation can be computed analytically for geometric Asian options while, for arithmetic ones, numerical methods must be applied.

2.3 Australian options

In this subsection we describe Australian options in discrete- and continuous-time, assuming either geometric or arithmetic average. We also present a relationship between Asian and Australian options and the put-call parity for Australian options.

Australians options were introduced in Moreno and Navas (2008). Their underlying asset is the ratio between the average price of an asset during the life option and the asset price at option maturity or, alternatively, the inverse of this ratio.

Similarly to Asian options, geometric or arithmetic averages can be considered in discrete- or continuous-time. Using equation (1), the expressions for geometric and arithmetic discrete-time average ratios are the following:

$$\frac{G^{(n)}}{P_n} = (P_n)^{-1} \cdot \prod_{i=1}^{n} P_i^{w_i} \tag{3}$$

$$\frac{P_n}{G^{(n)}} = P_n \cdot \prod_{i=1}^{n} P_i^{-w_i} \tag{4}$$

$$\frac{A^{(n)}}{P_n} = (P_n)^{-1} \cdot \sum_{i=1}^{n} w_i P_i \tag{5}$$

$$\frac{P_n}{A^{(n)}} = P_n \cdot \left[\sum_{i=1}^{n} w_i P_i\right]^{-1} \tag{6}$$

In a similar way, using equation (2), the expressions for geometric and arithmetic continuous-time average ratios are the following:

$$\frac{G^{(\infty)}}{P_T} = (P_T)^{-1} \cdot \exp\left(\int_t^T f(s) \ln(P_s)\, ds\right) \tag{7}$$

$$\frac{P_T}{G^{(\infty)}} = P_T \cdot \exp\left(-\int_t^T f(s) \ln(P_s)\, ds\right) \tag{8}$$

$$\frac{A^{(\infty)}}{P_T} = (P_T)^{-1} \cdot \int_t^T f(s)(P_s)\, ds \tag{9}$$

$$\frac{P_T}{A^{(\infty)}} = P_T \cdot \left(\int_t^T f(s) P_s\, ds\right)^{-1} \tag{10}$$

The next Proposition states a relationship between Australian and Asian options.

Proposition 2.5. *For the ratios (3)-(5), Australian options are a particular case of Asian options.*

Proof. Consider the relative asset price $P_i^* = \dfrac{P_i}{P_n}$, $i = 1, \ldots, n$. Then, the ratios (3)-(5) can be expressed as (geometric or arithmetic) averages:

$$\frac{G^{(n)}}{P_n} = \prod_{i=1}^{n} (P_i^*)^{w_i}, \quad \frac{P_n}{G^{(n)}} = \prod_{i=1}^{n} (P_i^*)^{-w_i}, \quad \frac{A^{(n)}}{P_n} = \sum_{i=1}^{n} w_i P_i^*$$

Then, for these ratios, Australian options are a particular case of Asian options. ■

As with Asian options, we can also present a put-call parity for Australian options. Consider Australian options with strike X and maturity T being its underlying *Ratio* that denotes one of the ratios described in equations (3)-(10). The final pay-off for a call (resp., put) is the positive part of $Ratio - X$ (resp., $X - Ratio$).

Let $c_{AUS}(r_t, t, T; T_b)$ and $p_{AUS}(r_t, t, T; T_b)$ denote, respectively, the premiums at time t of both options. The following Proposition provides the put-call parity.

Proposition 2.6. *Under no-arbitrage opportunities, the relationship between the premiums of these Australian options is given by*

$$p_{AUS}(r_t, t, T; T_b) + P(r_t, t, T)\widetilde{\mathbb{E}}_t [Ratio] = c_{AUS}(r_t, t, T; T_b) + P(r_t, t, T)X$$

or, equivalently, $c_{AUS}(r_t, t, T; T_b) - p_{AUS}(r_t, t, T; T_b) = P(r_t, t, T) \left[\widetilde{\mathbb{E}}_t [Ratio] - X\right]$.

■

As we will see, the conditional expected value of the geometric ratio can be computed analytically. For arithmetic ratios, numerical methods will be required.

3 Models for interest rates

This section introduces the TSIR models that we will use to price Asian and Australian options. In short, we present the Moreno *et al.* (2018) model and a particular case of this model, namely, the specification proposed in Vasicek (1977).

Let r_t denote the instantaneous interest rate at time t. Moreno *et al.* (2018) assumed that the time evolution of r_t is given by the following stochastic differential equation

$$dr_t = k(f(t) - r_t)dt + \sigma dW_t \tag{11}$$

where $k, \sigma \in \mathbb{R}^+$ denote, respectively, the speed of mean-reversion and the volatility of the diffusion, and W_t is a standard Wiener process. In addition, the mean-reversion level, $f(t)$, follows a time-dependent process driven by a Fourier series.

$$f(t) = \sum_{m=0}^{\infty} Re\left[A_m e^{im\omega t}\right] \tag{12}$$

These authors consider only the real part of the Fourier series since it is the only part with economic sense. They mention that $A_m \in \mathbb{C}$ for all m, so there is a phase factor contained in A_m. In more detail, $A_m = A_{m,x} + iA_{m,y}$, where $A_{m,x}, A_{m,y} \in \mathbb{R}$ and denote, respectively, the amplitude and the phase of the fluctuations of the instantaneous interest rate. Finally, the parameter ω represents the time frequency.

This model nests that in Vasicek (1977) by taking $A_m = 0$ for $m \in \mathbb{N} - \{0\}$ in equation (12). In this case, equation (11) becomes

$$dr_t = k(\mu - r_t)dt + \sigma dW_t \tag{13}$$

where $\mu = A_0$ indicates the (constant) mean-reversion level at which interest rates converge.

Let $\Lambda(r_t, t)$ denote the market price of risk, which is assumed to be constant, $\Lambda(r_t, t) = \lambda$. Then, the risk-neutral version of the process (11) is given by

$$dr_t = k(\alpha + g(t) - r_t)dt + \sigma d\tilde{W}_t \tag{14}$$

with

$$\alpha = A_0 - \frac{\lambda\sigma}{k}$$
$$g(t) = f(t) - A_0 = \sum_{m=1}^{\infty} Re\left[A_m e^{im\omega t}\right]$$

where $A_0 \in \mathbb{R}$ and $\tilde{W}_t = W_t + \lambda t$ is a standard Wiener process under the risk-neutral measure $\tilde{P}$. Moreno *et al.* (2018) obtain the following Proposition that provides the solution of the stochastic differential equation (11).

Proposition 3.1. *The solution of the risk-neutral process (11) followed by the instantaneous interest rate is given by*

$$r_s = e^{-k(s-t)} r_t + (1 - e^{-k(s-t)})\alpha + \sum_{m=1}^{\infty} Re \left[\frac{kA_m}{k + im\omega} (e^{im\omega s} - e^{-k(s-t)+im\omega t}) \right]$$
$$+ \sigma \int_t^s e^{-k(s-u)} d\tilde{W}_u, \quad \text{for all } s > t$$

■

For illustrative purposes of the flexibility that this model can offer, Figure 1 (borrowed from Moreno *et al.* (2018)) shows the very different shapes that the TSIR can take under this model.

Proposition 3.1 shows that the instantaneous interest rate follows a conditional Gaussian distribution. Its first two statistical moments under the measure $\tilde{P}$ are

$$\widetilde{\mathbb{E}}_t[r_T] = e^{-k(T-t)} r_t + (1 - e^{-k(T-t)})\alpha + \sum_{m=1}^{\infty} Re \left[\frac{kA_m}{k + im\omega} (e^{im\omega T} - e^{-k(T-t)+im\omega t}) \right] \tag{15}$$

$$\widetilde{\mathbb{V}}_t[r_T] = H(2k, T-t)\, \sigma^2 \tag{16}$$

where

$$H(p, q) = \frac{1 - e^{-pq}}{p} \tag{17}$$

Let $P(r_t, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at maturity T. These authors get the pricing partial differential equation (PDE) that must be verified by the price of any derivative

$$\frac{1}{2}\sigma^2 P_{rr}(r_t, t, T) + k(\alpha + g(t) - r_t) P_r(r_t, t, T) + P_t(r_t, t, T) - r_t P(r_t, t, T) = 0 \tag{18}$$

For this zero-coupon bond, the terminal condition is $P(r_T, T, T) = 1$ for all r_T.

Using probabilistic techniques, this price can be written as a conditional expectation under $\tilde{P}$,

$$P(r_t, t, T) = \widetilde{\mathbb{E}} \left[e^{-\int_t^T r_s ds} | r_t \right]$$

Working on this expression these authors obtain the following Proposition that provides an exponential-affine functional form for bond pricing.

Proposition 3.2. *The price at time t of a zero-coupon bond with maturity T and that amortizes with face value \$1 is given by*

$$P(r_t, t, T) = e^{A(t,T) - B(t,T) r_t}$$

where

$$\begin{aligned} A(t,T) =& -\frac{\sigma^2}{4k} B^2(t,T) + r^*(B(t,T) - (T-t)) \\ &- \sum_{m=1}^{\infty} Re \left[\frac{k A_m}{m\omega(k + im\omega)} (e^{im\omega t}(i - m\omega B(t,T)) - i e^{im\omega T}) \right] \\ B(t,T) =& H(k, T-t) \end{aligned}$$

where $H(\cdot,\cdot)$ is given by (17) and $r^* = \alpha - \frac{\sigma^2}{2k^2}$. ■

Remark 3.3. *Note that taking $A_m = 0$, $\forall m \in \mathbb{N} - \{0\}$ in expression (11) we obtain the Vasicek (1977) model. Then, we can obtain the analogous results presented in this section for the Vasicek (1977) model just eliminating the mentioned values for the A_m coefficients.* ■

4 Pricing of discrete-time options

In this Section, we will value Asian and Australian calLs where the average bond price is computed in discrete-time. We will consider either geometric and arithmetic averages and will obtain closed-form expressions for the premiums of geometric options under the Moreno *et al.* (2018) model. Later, applying Remark 3.3, the corresponding formula for the Vasicek (1977) model immediately arises. Arithmetic-discrete options will be valued numerically by Monte Carlo simulations.

Proposition 4.1. *Let $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})' \in \mathbb{R}^{n \times 1}$ be a random vector whose elements are the instantaneous interest rate at different times T_i, $i = 1, \ldots, n$. Then, the conditional distribution of R_d given r_t, with $t < T_i$ for all i, is a multivariate Gaussian distribution. Specifically, the conditional mean, $\mu_{(n)}$, and variance-*

covariance matrix, $\Omega_{(n)}$, are given by

$$\mu_{(n)} = \left(\widetilde{\mathbb{E}}_t[r_{T_1}], \ldots, \widetilde{\mathbb{E}}_t[r_{T_n}]\right)' \in \mathbb{R}^{n\times 1}$$

$$\Omega_{(n)} = \begin{pmatrix} \widetilde{\mathbb{V}}_t(r_{T_1}) & \widetilde{Cov}_t(r_{T_1}, r_{T_2}) & \cdots & \widetilde{Cov}_t(r_{T_1}, r_{T_n}) \\ \widetilde{Cov}_t(r_{T_1}, r_{T_2}) & \widetilde{\mathbb{V}}_t(r_{T_2}) & \cdots & \widetilde{Cov}_t(r_{T_2}, r_{T_n}) \\ \vdots & & \ddots & \vdots \\ \widetilde{Cov}_t(r_{T_1}, r_{T_n}) & \cdots & \widetilde{Cov}_t(r_{T_{n-1}}, r_{T_n}) & \widetilde{\mathbb{V}}_t(r_{T_n}) \end{pmatrix} \in \mathbb{R}^{n\times n}$$

where, for all $i = 1, \ldots, n$, $\widetilde{\mathbb{E}}_t[r_{T_i}]$ and $\widetilde{\mathbb{V}}_t(r_{T_1})$ are given by (15) and (16) respectively and

$$\widetilde{Cov}_t(r_{T_i}, r_{T_j}) = u(t, T_i, T_j) = \frac{\sigma^2}{2k} e^{-k(T_i+T_j)}(e^{2kT_i} - e^{2kt}), \quad T_i < T_j$$

Proof. See Technical Appendix. ∎

Using the equivalent definition for the multivariate Gaussian distribution given in Proposition 2.3 we obtain the next result.

Corollary 4.2. *Let $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})' \in \mathbb{R}^{n\times 1}$ be a random vector whose elements are the instantaneous interest rate at times T_i, $i = 1, \ldots, n$. Then, given r_t, for all $a \in \mathbb{R}^{n\times 1}$, the linear combination $a'R_d$ follows a conditional univariate Gaussian distribution with mean $a'\mu_{(n)}$ and variance $a'\Omega_{(n)}a$, where $\mu_{(n)}$ and $\Omega_{(n)}$ are given by Proposition 4.1.* ∎

This result shows that the variables $r_{T_1}, \ldots, r_{T_n}$ do not need to be independent to find the distribution of a linear combination of these random variables.

Now we can present in the next Proposition the closed-form expression for the value of any derivative whose final pay-off depends on a linear combination of the interest rate at different times, when the instantaneous interest rate follows the process given by (11).

Proposition 4.3. *Assume that the interest rate evolves as given by (11). Let $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})' \in \mathbb{R}^{n\times 1}$ be a random vector whose elements are the instantaneous interest rates at times T_i, $i = 1, \ldots, n$ and $a \in \mathbb{R}^{n\times 1}$. Then, given r_t, the value at time t of any interest rate derivative with terminal pay-off $U(a'R_d)$ is given by*

$$U(r_t, t, T) = P(r_t, t, T)\widetilde{\mathbb{E}}[U(r')|r_t], \quad r' \sim N(M_{(n)} - Q_{(n)}, V^2_{(n)})$$

where

$$\begin{aligned} M_{(n)} \equiv& M_{(n)}(r_t, t, T) = a' \mu_{(n)} \\ V_{(n)}^2 \equiv& V_{(n)}^2(t, T) = a' \; \Omega_{(n)} \; a \\ Q_{(n)} \equiv& Q_{(n)}(t, T) = \frac{\sigma^2}{2} \; a' h_{(n)} \end{aligned}$$

where $\mu_{(n)}$ and $\Omega_{(n)}$ are given by Proposition 4.1 and $h_{(n)} = (H^2(k, T_1 - t), \ldots, H^2(k, T_n - t))'$ $\in \mathbb{R}^{n \times 1}$, with $H(\cdot, \cdot)$ given by (17).

Proof. See Technical Appendix. ∎

Remark 4.4. *Note that taking $R_d = r_{T_n}$ and $a = 1$ we obtain the general pricing formula for a derivative whose final pay-off is a function of the interest rate at option maturity. Then, this result generalizes the theory in Moreno et al. (2018).* ∎

4.1 Pricing of geometric Asian calls

Working on the expression (1), we can express $G^{(n)} = e^{C_{(n)} - b'_{(n)} R_d}$ where $C_{(n)} = \sum_{i=1}^{n} w_i A(T_i, T_b)$, $b_{(n)} = (w_1 B(T_1, T_b), \ldots, w_n B(T_n, T_b))'$ and $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})'$. Let denote $\widetilde{G}^{(n)} = e^{C_{(n)} - r'}$, with r' following the conditional distribution given in Proposition 4.3, with $a = b_{(n)}$. Then, $\widetilde{G}^{(n)}$ follows a conditional lognormal distribution with $\widetilde{\mathbb{E}}_t \left[\widetilde{G}^{(n)}\right] = e^{E_{(n)} + \frac{1}{2} V_{(n)}^2}$, where $E_{(n)} = C_{(n)} - M_{(n)} + Q_{(n)}$, with $M_{(n)}$, $V_{(n)}^2$, and $Q_{(n)}$ as given in Proposition 4.3.

Now the Asian call premium is obtained using Proposition 4.3.

Proposition 4.5. *Consider an Asian call option with strike X and maturity T. The underlying is the geometric-discrete average of the price of a zero-coupon bond with maturity $T_b > T$. If the time evolution of the instantaneous interest rate follows the process (11), then the premium at time t of this call is given by*

$$c_A(r_t, t, T; T_b) = P(r_t, t, T) \left[\widetilde{\mathbb{E}}_t \left[\widetilde{G}^{(n)}\right] N(d_1) - X N(d_2)\right]$$

where $P(r_t, t, T)$ is given by Proposition 3.2 and

$$d_1 = \frac{\ln\left(\frac{\widetilde{\mathbb{E}}_t\left[\widetilde{G}^{(n)}\right]}{X}\right) + \frac{1}{2} V_{(n)}^2}{V_{(n)}}, \quad d_2 = d_1 - V_{(n)}$$

Proof. See Technical Appendix. ■

Table 1 provides the premiums of geometric Asian calls using the Moreno *et al.* (2018) and Vasicek (1977) models.[3] In both models, we consider different values for the parameters that affect the option premium and several number of monitoring dates n. Obviously, the case $n = \infty$ corresponds to continuous-time calls, analyzed and priced in the next Section.

To complement this information, Figures 2 to 6 show, respectively, how the premium of geometric-discrete Asian calls changes as a function of the TSIR parameters (speed of mean-reversion, constant term in the long-term value (μ or α), and diffusion coefficient), the option maturity, and the initial instantaneous interest rate. These Figures are computed for $n = 100$ but we have also computed these Figures for geometric-continuous calls and the results are qualitatively the same. These Figures are available upon request.

If the mean-reversion level is higher than the initial interest rate, we can see the following features:

- A higher value in the speed of mean-reversion implies a lower premium in Asian calls. The reason is twofold: a) there is a lower uncertainty in interest rates and hence in the zero-coupon bond price and its average and b) interest rates tend to higher values and this leads to a lower price of the zero-coupon bond and, then, in the average bond price.

- When the constant term in the level of mean-reversion increases, the option premium also decreases, converging quickly to zero. This is because interest rates tend to higher values and, then, the price of the zero-coupon bond decreases, implying a decrease in the average bond price and, then, in the option premium.

- Option premium increases with the diffusion coefficient. This is because this parameter affects positively to the variance of the interest rates and hence, there is more uncertainty in the price of the zero-coupon bond and its average.

- A higher option maturity implies two effects:

[3] For simplicity, all the Tables and Figures are based on the simplest weights, that is, $w_i = 1/n, \forall i$ and $f(s) = 1/(T-t)$, $\forall s \in [t, T]$ for, respectively, discrete- and continuous-time options.

i) First, there is more uncertainty in the price of the zero-coupon along the life of the option and this leads to a higher option premium.

ii) Depending on the initial parameters, $P(r_t, t, T)$ may decrease and this would lead to a lower option premium.

As a consequence, the premium of Asian calls can increase or decrease with the option maturity depending on the relative importance of these two effects.

- Finally, a higher initial instantaneous interest rate implies a lower premium of Asian calls because the discount factor, the zero-coupon bond price, and the average bond price reach lower values.

Conversely, if the mean-reversion level is lower than the initial interest rate, the effect of changing these parameters on these call premiums is qualitatively the same as before except for the speed of mean-reversion. In this case, a higher speed of mean-reversion can imply higher premiums. This is because, as before, there is lower uncertainty in interest rates but now the mean-reversion value can increase the bond price along the option life, and, then, the average bond price.

4.2 Pricing of geometric Australian calls

As in Section 4.1, we can express $\dfrac{G^{(n)}}{P_n} = e^{C^*_{(n)} - b^{*\prime}_{(n)} R_d}$A and $\dfrac{P_n}{G^{(n)}} = e^{-C^*_{(n)} + b^{*\prime}_{(n)} R_d}$, where $C^*_{(n)} = \sum_{i=1}^{n} w_i A(T_i, T_b) - A(T_n, T_b)$, $b^*_{(n)} = (w_1 B(T_1, T_b), \ldots, \ (w_n - 1) B(T_n, T_b))'$ and $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})'$.

Let denote $\dfrac{\widetilde{G}^{(n)}}{\widetilde{P}_n} = e^{C*_{(n)} - r'}$ and $\dfrac{P_n}{G^{(n)}} = e^{-C*_{(n)} + r'}$, with r' following the conditional distribution given in Proposition 4.3 with $a = b^*_{(n)}$. These ratios follow a conditional lognormal distribution with conditional expectation given as

$$\widetilde{\mathbb{E}}_t \left[\frac{\widetilde{G}^{(n)}}{\widetilde{P}_n} \right] = e^{E^*_{(n)} + \frac{1}{2} V^{*2}_{(n)}}, \quad \widetilde{\mathbb{E}}_t \left[\frac{\widetilde{P}_n}{\widetilde{G}^{(n)}} \right] = e^{-E^*_{(n)} + \frac{1}{2} V^{*2}_{(n)}} \tag{19}$$

with $E^*_{(n)} = C^*_{(n)} - M^*_{(n)} + Q^*_{(n)}$, where $M^*_{(n)}$, $Q^*_{(n)}$, and $V^{*2}_{(n)}$ are given in Proposition 4.3 with $a = b^*_{(n)}$.

Since $\dfrac{\widetilde{G}^{(n)}}{\widetilde{P}_n}$, $\dfrac{\widetilde{G}^{(n)}}{\widetilde{P}_n}$, and $\widetilde{G}^{(n)}$ follows the same conditional distribution with different statistical moments, the Australian call premium can be obtained using Proposition 4.3 in the same way as in Proposition 4.5.

Proposition 4.6. *Consider a geometric-discrete Australian call with strike X and maturity T. The underlying can be the ratio $\dfrac{G^{(n)}}{P_n}$ or $\dfrac{P_n}{G^{(n)}}$. If the time evolution of the instantaneous interest rate follows the process (??), the premium at time t of this call is given by*

$$c_{AUS}(r_t, t, T; T_b) = P(r_t, t, T)\left[\widetilde{\mathbb{E}}_t[\cdot]N(d_1) - XN(d_2)\right]$$

where $P(r_t, t, T)$ is given by Proposition 3.2 and

$$d_1 = \frac{\ln\left(\dfrac{\widetilde{\mathbb{E}}_t[\cdot]}{X}\right) + \dfrac{1}{2}V_{(n)}^{*2}}{V_{(n)}^{*}}, \quad d_2 = d_1 - V_{(n)}^{*}$$

where $\widetilde{\mathbb{E}}_t[\cdot]$ represents the conditional expected value of the respective ratio, see (19).

■

Table 2 shows the premium of geometric Australian calls, using the Moreno *et al.* (2018) and Vasicek (1977) models. We consider different values for the parameters that affect the option premium and several number of monitoring dates n. In more detail, Figures 7 to 11 show, respectively, how the premium of geometric-discrete Australian calls change as a function of the speed of mean-reversion, the diffusion coefficient, the constant term in the level of mean-reversion, the option maturity, and the instantaneous interest rate at initial time.

In this case the effect of changes in the different parameters on the call premium depends on both the zero-coupon bond price at option maturity and its average along the option life. So, the current analysis can be different from the analysis provided in Subsection 4.1.

The main qualitative conclusions of Table 2 are as follows:

- Higher values in the speed of mean-reversion imply less uncertainty in the price of the zero-coupon along the life of the option. Also, when α increases, interest rates tend to higher values, then the price of a zero-coupon bond decreases and hence the average bond price decreases too.

This implies two effects on Australian calls:

i) The ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$ can increase or decrease, depending on the relative change in both numerator and denominator. Then the option premium can increase or decrease.

ii) Based on the initial parameters, $P(r_t, t, T)$ may decrease and this leads to a lower premium.

Then, the premium of Australian calls can increase or decrease (monotonically or not) with the level or the speed of mean-reversion depending on the relative importance of these two effects.

- As in Moreno and Navas (2008), we can see that the effect of a higher diffusion coefficient on Australian calls can be not monotonous. The reason is that, even there is more uncertainty in the price of the bond and in its average, the effect on the ratios of a higher diffusion coefficient changes for the different values of this coefficient.

- The effect of a higher option maturity can imply two effects:

  i) First, there is more uncertainty in the price of the zero-coupon bond along the life of the option and this leads to a higher call premium.

  ii) Depending on the initial parameters, $P(r_t, t, T)$ may decrease and this leads to a lower call premium.

  As a consequence, the premium of Australian calls can increase or decrease (monotonically or not) with respect to the option maturity depending on the relative importance of these two effects.

- Finally, a higher initial instantaneous interest rate implies two effects on the premium of Australian calls:

  i) The ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$ can increase or decrease, depending on the relative change in both numerator and denominator. Then the option premium can increase or decrease.

ii) The discount factor $P(r_t, t, T)$ decrease and this leads to a lower premium.

Then, the premium of Australian calls can increase or decrease as a function of the initial interest rate.

4.3 Pricing of arithmetic Asian calls

Table 3 includes the premiums of arithmetic Asian calls using the Moreno *et al.* (2018) and Vasicek (1977) models. We can see that, under the same conditions, the arithmetic premium is higher than the geometric premium. The reason is that the geometric average is a lower bound of the arithmetic average and, as we are dealing with calls, the higher the underlying value, the higher the option premium.

For the sake of brevity, we do not include Figures for these calls as they are qualitatively analogous to the geometric-discrete Asian calls, that is, the effect of the different parameters on the premiums of Asian calls is qualitatively the same for both arithmetic or geometric averages. This also happens for arithmetic Australian calls and then we skip all these Figures, being all of them available upon request.

4.4 Pricing of arithmetic Australian calls

As we do not have an analytical pricing expression for arithmetic Asian calls, we can not obtain an analytical expression for the premium of arithmetic Australian calls. Then, these options will be also valued numerically by means of Monte Carlo simulations.

Recall that, by Proposition 2.5, the arithmetic-discrete Australian call with ratio $A^{(n)}/P_n$ can be interpreted as Asian option but this is not the case for the Australian call with ratio $P_n/A^{(n)}$.

Table 4 provides the premiums of arithmetic Australian calls, using the Moreno *et al.* (2018) and Vasicek (1977) models. In this case, with the same conditions, the premium of the call for the ratio $A^{(n)}/P_n$ is higher than that of the equivalent geometric call (see Table 2), and the opposite happens for the ratio $P_n/A^{(n)}$. As in Asian options, the reason is based on the comparison between arithmetic and geometric averages.

5 Pricing of continuous-time options

We will value now Asian and Australian call options in which the average bond price is computed in continuous-time, considering geometric and arithmetic averages. The distribution of the continuous-time arithmetic average is unknown and, then, it is approximated by the discrete-time arithmetic average with a very high value of n. As in the discrete-time case, Arithmetic-continuous options will be valued numerically by Monte Carlo simulations.

Then, we analyze now the geometric average case. As in Section 4, geometric call options will be valued analytically under the Moreno *et al.* (2018) model and we will apply Remark 3.3 to obtain closed-form expressions for the Vasicek (1977) model.

In this Section, the pricing methodology is based mainly in the fact that, similarly to the proof of Proposition 5.3, $\int_t^T g(s) r_s ds$ follows a Gaussian distribution with mean $\int_t^T g(s)\widetilde{\mathbb{E}}_t[r_s]\, ds$ and variance $\sigma^2 \int_t^T \left(\int_s^T g(s)e^{-ku}du\right)^2 e^{2ks}ds$, assuming that $\left(\int_s^T g(s)e^{-ku}du\right)^2$ is integrable in $[t,T]$.

Proposition 5.1. *Let $R_c = \left(\int_t^T g(s)r_s ds, r_T\right)'$ be a random vector with r_s the instantaneous interest rate at time s. Then, given r_t with $t < T$, R_c follows a conditional bivariate Gaussian distribution. Specifically, the conditional mean, $\mu_{(\infty)}$, and variance-covariance matrix, $\Omega_{(\infty)}$, are given by*

$$\mu_{(\infty)} = \left(\widetilde{\mathbb{E}}_t\left[\int_t^T g(s)r_s ds\right], \widetilde{\mathbb{E}}_t\left[r_T\right]\right)$$

$$\Omega_{(\infty)} = \begin{pmatrix} \widetilde{\mathbb{V}}_t\left(\int_t^T g(s)r_s ds\right) & \widetilde{Cov}_t\left(\int_t^T g(s)r_s ds, r_T\right) \\ \widetilde{Cov}_t\left(\int_t^T g(s)r_s ds, r_T\right) & \widetilde{\mathbb{V}}_t(r_T) \end{pmatrix}$$

with

$$\widetilde{\mathbb{E}}_t\left[\int_t^T g(s)r_s ds\right] = \int_t^T g(s)\widetilde{\mathbb{E}}_t[r_s]ds$$

$$\widetilde{\mathbb{V}}_t\left(\int_t^T g(s)r_s ds\right) = \sigma^2 \int_t^T \left(\int_s^T g(s)e^{-ku}du\right)^2 e^{2ks}ds$$

$$\widetilde{Cov}_t\left(\int_t^T g(s)r_s ds, r_T\right) = \sigma^2 \int_t^T e^{-k(T-2s)} \left(\int_s^T g(s)e^{-ku}du\right) ds$$

where $\widetilde{\mathbb{E}}_t[r_T]$ *and* $\widetilde{\mathbb{V}}_t(r_T)$ *are given by (15) and (16).* ■

Corollary 5.2. *Let* $R_c = \left(\int_t^T g(s) r_s ds, r_T\right)'$ *be a random vector with* r_s *the instantaneous interest rate at time s. Then, given* r_t, $t < s$, *for all* $a \in \mathbb{R}^{2\times 1}$, *the linear combination* $a' R_c$ *follows a conditional univariate Gaussian distribution with mean* $a'\mu_{(\infty)}$ *and variance* $a'\Omega_{(\infty)} a$, *where* $\mu_{(\infty)}$ *and* $\Omega_{(\infty)}$ *are given by Proposition 4.1.* ■

We present now the continuous-time version of Proposition 4.3.

Proposition 5.3. *Assume that the interest rate evolves as given by (11). Let* $R_c = \left(\int_t^T g(s) r_s ds, r_T\right)'$ *and* $a \in \mathbb{R}^{2\times 1}$. *Then, given* r_t, *the value at time t of any interest rate derivative with terminal pay-off* $U(a' R_c)$ *is given by*

$$U(r_t, t, T) = P(r_t, t, T)\widetilde{\mathbb{E}}[U(r')|r_t], \quad r' \sim N(M_{(\infty)} - Q_{(\infty)}, V^2_{(\infty)})$$

with

$$\begin{aligned} M_{(\infty)} &\equiv M_{(\infty)}(r_t, t, T) = a'\mu_{(\infty)} \\ Q_{(\infty)} &\equiv Q_{(\infty)}(t, T) = a' q_{(\infty)} \\ V^2_{(\infty)} &\equiv V^2_{(\infty)}(t, T) = a'\Omega_{(\infty)} a \end{aligned}$$

where $\mu_{(\infty)}$ *and* $\Omega_{(\infty)}$ *are given by Proposition 5.1, and*

$$q_{(\infty)} = \sigma^2 \left(\frac{1}{k} \int_t^T \left(e^{ks} - e^{-k(T-2s)}\right) \left(\int_s^T g(u) e^{-ku} du \right) ds, \frac{1}{2} H^2(k, Tb - t) \right)$$

with $H(\cdot,\cdot)$ *given by (17).*

Proof. See Technical Appendix. ■

5.1 Pricing of geometric Asian calls

Working on (2), we can express $G^{(\infty)} = e^{\int_t^T f(s)A(s,T_b)ds - \int_t^T f(s)B(s,T_b) r_s ds}$. Let $\widetilde{G}^{(\infty)} = e^{C_{(\infty)} - r'}$ where $C_{(\infty)} = \int_t^T f(s)A(s,T_b)ds$ and r' follows the conditional distribution given in Proposition 5.3 with $g(s) = f(s)B(s,T_b)$ and $a = (1,0)'$. Then, $\widetilde{G}^{(\infty)}$ follows a conditional lognormal distribution with conditional expectation,

$$\widetilde{\mathbb{E}}_t\left[\widetilde{G}^{(\infty)}\right] = e^{E_{(\infty)} + \frac{1}{2} V^2_{(\infty)}} \tag{20}$$

with $E_{(n)} = C_{(\infty)} - M_{(\infty)} + Q_{(\infty)}$, where $M_{(\infty)}$, $Q_{(\infty)}$, and $V^2_{(\infty)}$ are given by Proposition 5.3.

The next Proposition provides the premium of geometric-continuous Asian calls.

Proposition 5.4. *Consider an Asian call option with strike X and maturity T. The underlying is the geometric-continuous average of the price of a zero-coupon bond with maturity $T_b > T$. If the time evolution of the instantaneous interest rate follows the process (11), then the premium at time t of this call is given by*

$$c_A(r_t, t, T; T_b) = P(r_t, t, T)\left[\widetilde{\mathbb{E}}_t\left[\widetilde{G}^{(\infty)}\right] N(d_1) - XN(d_2)\right]$$

where $P(r_t, t, T)$ is given by Proposition 3.2 and

$$d_1 = \frac{\ln\left(\frac{\widetilde{\mathbb{E}}_t\left[\widetilde{G}^{(\infty)}\right]}{X}\right) + \frac{1}{2}V^2_{(\infty)}}{\Sigma_{(\infty)}}, \quad d_2 = d_1 - V_{(\infty)}$$

■

5.2 Pricing of geometric Australian calls

As in Section 4.1 we can express $\dfrac{G^{(\infty)}}{P_T} = e^{C^*_{(\infty)} - b^{*\prime}_{(\infty)} R_c}$ and $\dfrac{P_T}{G^{(\infty)}} = e^{-C^*_{(\infty)} + b^{*\prime}_{(\infty)} R_c}$, where $C^*_{(\infty)} = \int_t^T f(s)A(s, T_b)ds - A(T_n, T_b)$, $b^*_{(\infty)} = (-1, B(T, T_b))$ and $R_c = \left(\int_t^T f(s)B(s, T_b)r_s ds, r_T\right)'$.

Let denote $\dfrac{\widetilde{G}^{(\infty)}}{\widetilde{P}_n} = e^{C*_{(\infty)} - r'}$ and $\dfrac{P_T}{G^{(\infty)}} = e^{-C*_{(\infty)} + r'}$, where r' follows the conditional distribution given in Proposition 5.3, with $g(s) = f(s)B(s, T_b)$ and $a = b^*_{(\infty)}$. These ratios follow a conditional lognormal distribution with expectations given by

$$\widetilde{\mathbb{E}}_t\left[\frac{\widetilde{G}^{(\infty)}}{\widetilde{P}_T}\right] = e^{E^*_{(\infty)} + \frac{1}{2}V^{*2}_{(\infty)}}, \quad \widetilde{\mathbb{E}}_t\left[\frac{\widetilde{P}_T}{\widetilde{G}^{(\infty)}}\right] = e^{-E^*_{(\infty)} + \frac{1}{2}V^{*2}_{(\infty)}} \tag{21}$$

with $E^*_{(\infty)} = C^*_{(\infty)} - M^*_{(\infty)} + Q^*_{(\infty)}$, where $M^*_{(\infty)}$, $Q^*_{(\infty)}$, and $V^{*2}_{(\infty)}$ are given in Proposition 4.3, with $g(s) = f(s)B(s, T_b)$ and $a = b^*_{(\infty)}$.

Proposition 5.5. *Consider a geometric-continuous Australian call with strike X and maturity T. The underlying can be the ratio $\dfrac{G^{(\infty)}}{P_T}$ or $\dfrac{P_T}{G^{(\infty)}}$. If the time evolution* *of the instantaneous interest rate follows the process (11), the premium at time t of this call is given by*

$$c_{AUS}(r_t, t, T; T_b) = P(r_t, t, T)\left[\widetilde{\mathbb{E}}_t[\cdot]N(d_1) - XN(d_2)\right]$$

where $P(r_t, t, T)$ is given by Proposition 3.2 and

$$d_1 = \frac{\ln\left(\frac{\widetilde{\mathbb{E}}_t[\cdot]}{X}\right) + \frac{1}{2}V^{*2}_{(\infty)}}{V^{*}_{(\infty)}}, \quad d_2 = d_1 - V^{*}_{(\infty)}$$

where $\widetilde{\mathbb{E}}_t[\cdot]$ represents the conditional expected value of the respective ratio, see (21).

■

As we mentioned in Subsections 4.1 and 4.2, Tables 1 and 2 show the premiums of, respectively, geometric Asian and Australian calls under the Moreno *et al.* (2018) and Vasicek (1977) models. Then, the effect of changes of the different parameters on these premiums have qualitatively the same interpretations as those indicated in these Subsections.

6 Conclusions

We have priced Asian and Australian calls on zero-coupon bonds assuming that the time evolution of interest rate is given by the model proposed in Moreno *et al.* (2018). This model generalizes that proposed in Vasicek (1977), introduces a large flexibility in interest rates, and may allow for a better empirical behavior while maintaining the analytical tractability. These authors achieve this assuming that the long-term value of interest rates is given by a Fourier series.

For both models, we have obtained analytical expressions for the premium of geometric calls while arithmetic calls have been valued by Monte Carlo simulations.

We have seen that, in both models, when we increase the constant term in the mean-reversion level or the initial instantaneous interest rate, the premium of the Asian call decreases. A higher speed of mean-reversion can imply a lower (higher) premium of the Asian calls if the mean-reversion level is higher (lower) than the initial interest rate. If the diffusion coefficient increases, the Asian call premium also increases. Finally, an increase in the option maturity can increase or decrease the Asian call premium.

Finally, under the Moreno *et al.* (2018) model, we have also shown that the effect of changes in the different parameters on Australian call premiums is very different. A higher constant term of the mean-reversion level or a higher initial instantaneous interest rate decrease the zero-coupon bond price and, then, its average price but - depending on the relative change in both numerator and denominator in the underlying ratio - the premium of these options can increase or decrease. Also, the Australian call premium can be a non-monotonous function of either the constant term in the diffusion coefficient, the speed of mean-reversion, or the option maturity.

A possible future line of research could be to analyze the evolution of interest rates using the Cox *et al.* (1985) model assuming that the level of mean-reversion is given by a Fourier series. Under this assumption, interest rates do not follow a chi-square distribution as they can take negative values. Then, we should obtain its probability distribution and a pricing expression for any derivative on a zero-coupon bond. We also leave for further research the pricing under these models of other exotic derivatives as, for example, compound, look-back, and barrier options.

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Technical Appendix

Proof of Proposition 4.1

Given $r_t \in \mathbb{R}$ and $t < T_i \leq T_j$, we have

$$\begin{aligned}\widetilde{Cov}_t(r_{T_i}, r_{T_j}) =& \widetilde{\mathbb{E}}_t \left[\left(r_{T_i} - \widetilde{\mathbb{E}}_t[r_{T_i}] \right) \left(r_{T_j} - \widetilde{\mathbb{E}}_t[r_{T_j}] \right) \right] \\ =& \widetilde{\mathbb{E}}_t \left[\left(\sigma \int_t^{T_i} e^{-k(T_i-u)} d\widetilde{W}_u \right) \left(\sigma \int_t^{T_j} e^{-k(T_j-u)} d\widetilde{W}_u \right) \right]\end{aligned}$$

Applying the generalized Itô isometry, we get

$$\widetilde{Cov}_t(r_{T_i}, r_{T_j}) = \sigma^2 e^{-k(T_i+T_j)} \widetilde{\mathbb{E}}_t \left[\int_t^{T_i} e^{2ku} du \right] = \frac{\sigma^2}{2k} e^{-k(T_i+T_j)} (e^{2kT_i} - e^{2kt})$$

So, if we consider the variance-covariance matrix of the different r_{T_i} values, Ω, this matrix is non-singular and hence applying Definition 2.2 concludes the proof.

Proof of Proposition 4.3

Let $R_d = (r_{T_1}, r_{T_2}, \ldots, r_{T_n})' \in \mathbb{R}^{n\times 1}$ be a random vector whose elements are the instantaneous interest rate at different times T_i, $i = 1, \ldots, n.$, $a \in \mathbb{R}^{n\times 1}$, and let $Y(t,s) = \int_t^s r_u du$. The solution of the PDE (18), with terminal condition $g(a'R_d)$, is given by

$$U(r_t, t, T) = \mathbb{E}_t[e^{-Y(t,T)} g(a'R_d)]$$

Similarly to Brigo and Mercurio (2001), we can expresss

$$\begin{aligned}Y(t,s) =& H(k, s-t) r_t - (H(k, s-t) - (s-t))\alpha \\ &+ \sum_{m=1}^{\infty} Re \left[\frac{e^{im\omega t}(m\omega e^{-k(s-t)} + ik - m\omega) - ike^{im\omega s}}{m\omega(k + im\omega)} A_m \right] + \sigma \int_t^s \frac{1 - e^{-k(s-u)}}{k} dW_u\end{aligned}$$

Then, $Y(t,s)$ follows a conditional Gaussian distribution with mean M_Y and variance

V_Y^2, given by

$$M_Y = H(k, s-t)r_t - (H(k, s-t) - (s-t))\alpha \\ + \sum_{m=1}^{\infty} Re\left[\frac{e^{im\omega t}(m\omega e^{-k(s-t)} + ik - m\omega) - ike^{im\omega s}}{m\omega(k+im\omega)} A_m\right] \tag{22}$$

$$V_Y^2 = \frac{\sigma^2}{k^2}\left[(s-t) - 2H(k, s-t) + H(2k, s-t)\right] \tag{23}$$

Using the generalized Itô isometry, we get $\widetilde{Cov}_t\left(Y(t,s), a'R_d\right) = \frac{\sigma^2}{2}a'h_{(n)}$ with $h_{(n)} = (H^2(k, T_1 - t), \ldots, H^2(k, T_n - t))$. Then, the random vector $X = (a'R_d, Y(t,s))'$ follows a bivariate Gaussian distribution with mean ξ and variance Σ, given by

$$\xi = (M_{(n)}, M_Y)', \quad \Sigma = \begin{pmatrix} V_{(n)}^2 & Q_{(n)} \\ Q_{(n)} & V_Y^2 \end{pmatrix}$$

where $Q_{(n)} = \widetilde{Cov}_t\left(Y(t,s), a'R_d\right)$, and $M_{(n)} = \widetilde{\mathbb{E}}_t[a'R_d]$ and $V_{(n)}^2 = \widetilde{\mathbb{V}}_t[a'R_d]$ are given in Corollary 4.2. Let $p(r_t, t, s, \cdot, \cdot)$ be the conditional density function of X. Then

$$U(r_t, t, T) = \int_{-\infty}^{\infty} G(r_t, r', t, T)g(r')dr', \quad G(r_t, r', t, T) = \int_{-\infty}^{\infty} e^{-y}p(r_t, t, s, r', y)dy \tag{24}$$

Substituting the expression of $p(r_t, t, s, r', y)$ in equation (24), some algebra leads to $G(r_t, r', t, T) = e^{\frac{1}{2}V_Y^2 - M_Y} f_1(r')$, where $f_1(\cdot)$ is the density function of a Gaussian variable with mean $M_{(n)} - Q_{(n)}$ and variance $V_{(n)}^2$. Then

$$U(r_t, t, T) = e^{\frac{1}{2}V_Y^2 - M_Y} \int_{-\infty}^{\infty} f_1(r')g(r')dr' = P(r_t, t, T)\widetilde{\mathbb{E}}_t[g(r')]$$

Proof of Proposition 4.5

$$
\begin{aligned}
c_A(r_t, t, T; T_b) &= P(r_t, t, T) E\left[\left(G^{(n)} - X\right)^+ \mid r_t\right] \\
&= P(r_t, t, T) \int_{-\infty}^{\infty} \left(e^{E_{(n)} + \Sigma_{(n)} z} - X\right)^+ f(z)\, dz \\
&= P(r_t, t, T) \int_{-\infty}^{d_2} \left(e^{E_{(n)} - \Sigma_{(n)} z} - X\right) \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2} z^2}\, dz \\
&= P(r_t, t, T) \left(\int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}}\, e^{E_{(n)} - \Sigma_{(n)} z - \frac{1}{2} z^2}\, dz - X \int_{\infty}^{d_2} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2} z^2}\, dz\right) \\
&= P(r_t, t, T) \left(e^{E_{(n)} + \frac{1}{2} \Sigma^2_{(n)}} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(z + \Sigma_{(n)})^2}\, dz - X N(d_2)\right) \\
&= P(r_t, t, T) \left(E\big[G^{(n)}\big] \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} u^2}\, du - X N(d_2)\right) \\
&= P(r_t, t, T) \left(E\left[G^{(n)}\right] N(d_1) - X N(d_2)\right)
\end{aligned}
$$

Proof of Proposition 5.3

Let $R_c = \left(\int_t^T g(s) r_s ds, r_T\right)'$ be a random vector with r_s the instantaneous interest rate at time s, $a \in \mathbb{R}^{2\times 1}$, and let $Y(t, s) = \int_t^s r_u du$. The solution of the PDE (18), with terminal condition $g(a' R_c)$, is given by

$$
U(r_t, t, T) = \mathbb{E}_t[e^{-Y(t,T)} g(a' R_c)]
$$

In the proof of Proposition 4.3 we saw that $Y(t, s)$ follows a conditional Gaussian distribution with mean M_Y and variance V_Y^2, given in expressions (22)-(23).

Using the generalized Itô isometry, we get $\widetilde{Cov}_t\left(Y(t, s), a' R_c\right) = \frac{\sigma^2}{2} a' h_{(\infty)}$ with $h_{(\infty)} = \sigma^2 \left(\frac{1}{k} \int_t^T \left(e^{ks} - e^{-k(T-2s)}\right) \left(\int_s^T g(u) e^{-ku} du\right) ds, \frac{1}{2} H^2(k, Tb - t)\right)$. Then, the random vector $X = (a' R_c, Y(t, s))'$ follows a bivariate Gaussian distribution with mean ξ and variance Σ, given by

$$
\xi = (M_{(\infty)}, M_Y)', \quad \Sigma = \begin{pmatrix} V^2_{(\infty)} & Q_{(\infty)} \\ Q_{(\infty)} & V_Y^2 \end{pmatrix}
$$

where $Q_{(\infty)} = \widetilde{Cov}_t\,(Y(t,s), a'R_c)$, and $M_{(\infty)} = \widetilde{\mathbb{E}}_t[a'R_c]$ and $V^2_{(\infty)} = \widetilde{\mathbb{V}}_t[a'R_c]$ are given in Corollary 4.2. Let $p(r_t, t, s, \cdot, \cdot)$ be the conditional density function of X. Then

$$U(r_t, t, T) = \int_{-\infty}^{\infty} G(r_t, r', t, T) g(r') dr', \quad G(r_t, r', t, T) = \int_{-\infty}^{\infty} e^{-y} p(r_t, t, s, r', y) dy \tag{25}$$

Substituting the expression of $p(r_t, t, s, r', y)$ in equation (25), some algebra leads to $G(r_t, r', t, T) = e^{\frac{1}{2}V_Y^2 - M_Y} f_1(r')$, where $f_1(\cdot)$ is the density function of a Gaussian variable with mean $M_{(\infty)} - Q_{(\infty)}$ and variance $V^2_{(\infty)}$. Then

$$U(r_t, t, T) = e^{\frac{1}{2}V_Y^2 - M_Y} \int_{-\infty}^{\infty} f_1(r') g(r') dr' = P(r_t, t, T)\widetilde{\mathbb{E}}_t[g(r')]$$

Appendix of Tables

				Moreno *et al.* (2018) model					**Vasicek (1977) model**				
Parameters				Number of monitoring dates (n)					Number of monitoring dates (n)				
k	μ	σ	T	1	10	100	1000	∞	1	10	100	1000	∞
0.2	0.05	0.002	10	0.1217	0.0776	0.0740	0.0736	0.0736	0.1213	0.0771	0.0735	0.0732	0.0731
0.8	0.05	0.002	10	0.1059	0.0598	0.0559	0.0555	0.0555	0.1057	0.0596	0.0558	0.0554	0.0553
1.2	0.05	0.002	10	0.1045	0.0586	0.0547	0.0543	0.0543	0.1044	0.0585	0.0546	0.0542	0.0542
0.2	0.03	0.002	10	0.2739	0.2249	0.2206	0.2202	0.2201	0.2730	0.2239	0.2195	0.2191	0.2190
0.2	0	0.002	10	0.7248	0.7027	0.6995	0.6992	0.6992	0.7224	0.6998	0.6965	0.6961	0.6961
0.2	0.05	0.01	10	0.1282	0.0833	0.0796	0.0793	0.0792	0.1266	0.0822	0.0786	0.0783	0.0782
0.2	0.05	0.02	10	0.1497	0.1022	0.0984	0.0980	0.0979	0.1440	0.0992	0.0956	0.0952	0.0952
0.2	0.05	0.002	20	0.1746	0.0851	0.0786	0.0780	0.0779	0.1741	0.0847	0.0782	0.0776	0.0775
0.2	0.05	0.002	25	0.1934	0.0849	0.0775	0.0768	0.0767	0.1928	0.0846	0.0772	0.0765	0.0764

Table 1: Premium of geometric Asian calls using both the Moreno *et al.* (2018) and Vasicek (1977) models. We consider $X = 0.2$ and $T_b = 30$. In both models, $\lambda = 0$, $r_0 = 0.02$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

					Moreno *et al.* (2018) model					Vasicek (1977) model				
Parameters				Ratio	Number of monitoring dates (n)					Number of monitoring dates (n)				
k	α/μ	σ	T		1	10	100	1000	∞	1	10	100	1000	10000
0.2	0.05	0.002	10	$G^{(n)}/P_n$	0.5540	0.4367	0.4272	0.4262	0.4261	0.5525	0.4387	0.4286	0.4277	0.4277
				$P_n/G^{(n)}$	0.5540	0.6954	0.7095	0.7109	0.7111	0.5525	0.6889	0.7036	0.7050	0.7050
0.8	0.05	0.002	10	$G^{(n)}/P_n$	0.5055	0.3800	0.3695	0.3685	0.3684	0.5038	0.3834	0.3727	0.3717	0.3716
				$P_n/G^{(n)}$	0.5055	0.6620	0.6787	0.6804	0.6806	0.5038	0.6526	0.6693	0.6709	0.6711
1.2	0.05	0.002	10	$G^{(n)}/P_n$	0.4993	0.3744	0.3638	0.3627	0.3626	0.4975	0.3778	0.3670	0.3660	0.3659
				$P_n/G^{(n)}$	0.4993	0.6554	0.6724	0.6741	0.6743	0.4975	0.6458	0.6627	0.6644	0.6645
0.2	0.03	0.002	10	$G^{(n)}/P_n$	0.6207	0.5321	0.5243	0.5235	0.5234	0.6190	0.5321	0.5242	0.5233	0.5231
				$P_n/G^{(n)}$	0.6207	0.7207	0.7308	0.7318	0.7320	0.6190	0.7169	0.7270	0.7283	0.7284
0.2	0	0.002	10	$G^{(n)}/P_n$	0.7359	0.7137	0.7104	0.7101	0.7101	0.7339	0.7112	0.7083	0.7073	0.7072
				$P_n/G^{(n)}$	0.7359	0.7588	0.7622	0.7625	0.7626	0.7339	0.7573	0.7603	0.7613	0.7614
0.2	0.05	0.01	10	$G^{(n)}/P_n$	0.5566	0.4416	0.4322	0.4313	0.4312	0.5551	0.4441	0.4355	0.4358	0.4344
				$P_n/G^{(n)}$	0.5566	0.6971	0.7111	0.7125	0.7127	0.5551	0.6897	0.7025	0.7019	0.7043
0.2	0.05	0.02	10	$G^{(n)}/P_n$	0.5646	0.4572	0.4484	0.4476	0.4476	0.5630	0.4657	0.4551	0.4548	0.4548
				$P_n/G^{(n)}$	0.5646	0.7025	0.7162	0.7175	0.7179	0.5630	0.6863	0.7019	0.7030	0.7032
0.2	0.05	0.002	20	$G^{(n)}/P_n$	0.3422	0.1950	0.1844	0.1833	0.1832	0.3412	0.2040	0.1926	0.1915	0.1914
				$P_n/G^{(n)}$	0.3422	0.5668	0.5926	0.5952	0.5955	0.3412	0.5437	0.5695	0.5719	0.5721
0.2	0.05	0.002	25	$G^{(n)}/P_n$	0.2671	0.1278	0.1183	0.1174	0.1173	0.2663	0.1385	0.1282	0.1273	0.1272
				$P_n/G^{(n)}$	0.2671	0.5060	0.5353	0.5383	0.5386	0.2663	0.4736	0.5021	0.5048	0.5051

Table 2: Premium of geometric Australian calls using both the Moreno *et al.* (2018) and Vasicek (1977) models. We consider $X = 0.2$ and $T_b = 30$. In both models, $\lambda = 0$, $r_0 = 0.02$, $\sigma = 0.002$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

				Moreno *et al.* **(2018) model**					**Vasicek (1977) model**				
Parameters				Number of monitoring dates (n)					Number of monitoring dates (n)				
k	μ	σ	T	1	10	100	1000	∞	1	10	100	1000	∞
0.2	0.05	0.002	10	0.1218	0.0789	0.0752	0.0749	0.0748	0.1214	0.0785	0.0748	0.0744	0.0740
0.8	0.05	0.002	10	0.1059	0.0616	0.0577	0.0573	0.0572	0.1057	0.0614	0.0575	0.0571	0.0571
1.2	0.05	0.002	10	0.1045	0.0605	0.0565	0.0561	0.0561	0.1044	0.0604	0.0564	0.0560	0.0560
0.2	0.03	0.002	10	0.2741	0.2261	0.2215	0.2212	0.2211	0.2728	0.2251	0.2205	0.2201	0.2201
0.2	0	0.002	10	0.7248	0.7033	0.6993	0.6996	0.6991	0.7218	0.6999	0.6968	0.6960	0.6954
0.2	0.05	0.01	10	0.1286	0.0846	0.0804	0.0798	0.0792	0.1268	0.0839	0.0802	0.0796	0.0794
0.2	0.05	0.02	10	0.1502	0.1026	0.0985	0.0983	0.0983	0.1449	0.1009	0.0979	0.0977	0.0975
0.2	0.05	0.002	20	0.1748	0.0910	0.0840	0.0834	0.0833	0.1740	0.0906	0.0837	0.0831	0.0830
0.2	0.05	0.002	25	0.1935	0.0936	0.0856	0.0849	0.0847	0.1928	0.0933	0.0853	0.0845	0.0845

Table 3: Premium of arithmetic Asian calls using both the Moreno *et al.* (2018) and Vasicek (1977) models. We consider $X = 0.2$ and $T_b = 30$. In both models, $\lambda = 0$, $r_0 = 0.02$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

					Moreno *et al.* (2018) model					**Vasicek (1977) model**				
Parameters				Ratio	Number of monitoring dates (n)					Number of monitoring dates (n)				
k	α/μ	σ	T		1	10	100	1000	∞	1	10	100	1000	10000
0.2	0.05	0.002	10	$A^{(n)}/P_n$	0.5525	0.4387	0.4286	0.4277	0.4277	0.5540	0.4403	0.4305	0.4298	0.4297
				$P_n/A^{(n)}$	0.5525	0.6889	0.7036	0.7050	0.7050	0.5540	0.6903	0.7045	0.7056	0.7058
0.8	0.05	0.002	10	$A^{(n)}/P_n$	0.5038	0.3834	0.3727	0.3717	0.3716	0.5055	0.3850	0.3743	0.3732	0.3731
				$P_n/A^{(n)}$	0.5038	0.6526	0.6693	0.6709	0.6711	0.5055	0.6544	0.6710	0.6728	0.6730
1.2	0.05	0.002	10	$A^{(n)}/P_n$	0.4975	0.3778	0.3670	0.3660	0.3659	0.4993	0.3794	0.3687	0.3676	0.3675
				$P_n/A^{(n)}$	0.4975	0.6458	0.6627	0.6644	0.6645	0.4993	0.6476	0.6645	0.6662	0.6663
0.2	0.03	0.002	10	$A^{(n)}/P_n$	0.6190	0.5321	0.5242	0.5233	0.5231	0.6207	0.5342	0.5262	0.5254	0.5251
				$P_n/A^{(n)}$	0.6190	0.7169	0.7270	0.7283	0.7284	0.6207	0.7181	0.7284	0.7293	0.7298
0.2	0	0.002	10	$A^{(n)}/P_n$	0.7339	0.7112	0.7083	0.7073	0.7072	0.7359	0.7138	0.7109	0.7102	0.7105
				$P_n/A^{(n)}$	0.7339	0.7573	0.7603	0.7613	0.7614	0.7359	0.7586	0.7617	0.7625	0.7621
0.2	0.05	0.01	10	$A^{(n)}/P_n$	0.5551	0.4441	0.4355	0.4358	0.4344	0.5566	0.4474	0.4377	0.4368	0.4366
				$P_n/A^{(n)}$	0.5551	0.6897	0.7025	0.7019	0.7043	0.5566	0.6887	0.7029	0.7043	0.7045
0.2	0.05	0.02	10	$A^{(n)}/P_n$	0.5630	0.4657	0.4551	0.4548	0.4548	0.5646	0.4666	0.4588	0.4593	0.4580
				$P_n/A^{(n)}$	0.5630	0.6863	0.7019	0.7030	0.7032	0.5646	0.6884	0.7006	0.7001	0.7019
0.2	0.05	0.002	20	$A^{(n)}/P_n$	0.3412	0.2040	0.1926	0.1915	0.1914	0.3422	0.2046	0.1933	0.1923	0.1921
				$P_n/A^{(n)}$	0.3412	0.5437	0.5695	0.5719	0.5721	0.3422	0.5453	0.5708	0.5731	0.5736
0.2	0.05	0.002	25	$A^{(n)}/P_n$	0.2663	0.1385	0.1282	0.1273	0.1272	0.2671	0.1389	0.1287	0.1277	0.1276
				$P_n/A^{(n)}$	0.2663	0.4736	0.5021	0.5048	0.5051	0.2671	0.4750	0.5034	0.5062	0.5067

Table 4: Premium of arithmetic Australian calls using both the Moreno *et al.* (2018) and Vasicek (1977) models. We consider $X = 0.2$ and $T_b = 30$. In both models, $\lambda = 0$, $r_0 = 0.02$, $\sigma = 0.002$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

Appendix of Figures

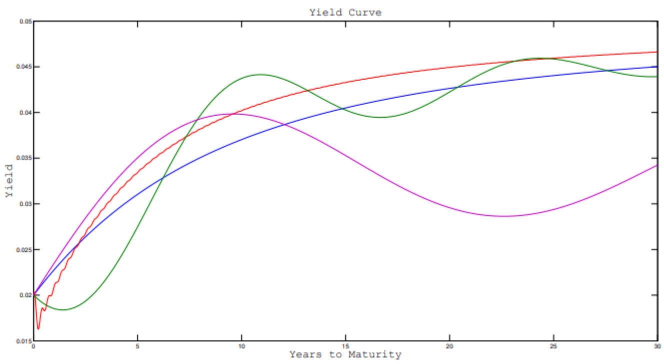

Figure 1: **Term structure of interest rates for arbitrary set of parameters in the Moreno *et al.* (2018) and Vasicek (1977) models.** In both models, $r_0 = 0.02$. For the Vasicek (1977) model (blue line), we consider $\mu = 0.05$, $\sigma = 0.002$ and $k = 0.2$. For the Moreno *et al.* (2018) model, we consider three alternatives: a) Red line: $\alpha = 0.05$, $\sigma = 0.0011$, $k = 0.3397$, $\omega = 20$, $n = 5$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$. b) Green line: $\alpha = 0.07$, $\sigma = 0.0005$, $k = 0.018$, $\omega = 0.48$, $n = 2$, $A_{1,x} = -1.8$, $A_{1,y} = 1$, $A_{2,x} = 1.5$, $A_{2,y} = -1.5$. c) Violet line: $\alpha = 0.08$, $\sigma = 0.0002$, $k = 0.02$, $\omega = 0.25$, $n = 1$, $A_{1,x} = 0.3$, $A_{1,y} = 0.03$.

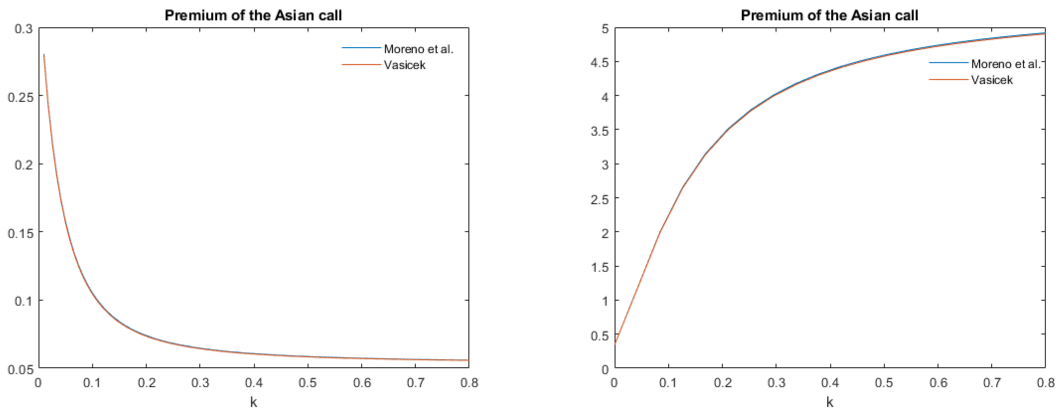

Figure 2: **Premium of geometric Asian calls as a function of the speed of mean-reversion in interest rates.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained with the Moreno *et al.* (2018) and Vasicek (1977) models, respectively. In both models, $\lambda = 0$, $r_0 = 0.02$, $\sigma = 0.002$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$. The first (resp., second) graph considers $\alpha = 0.05$ (resp., $\alpha = -0.05$) in the Moreno *et al.* (2018) model and $\mu = 0.05$ (resp., $\mu = -0.05$) in the Vasicek (1977) model.

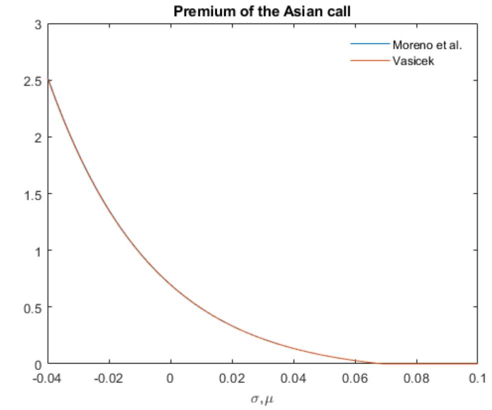

Figure 3: **Premium of geometric Asian calls as a function of the constant term in the level of mean-reversion in interest rates.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained with the Moreno *et al.* (2018) and Vasicek (1977) models, respectively. In both models, $\lambda = 0$, $r_0 = 0.02$, $k = 0.02$, $\sigma = 0.002$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

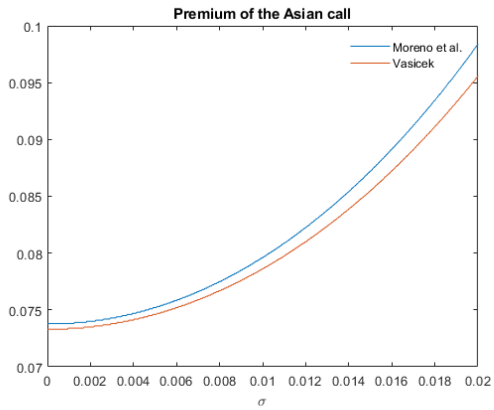

Figure 4: **Premium of geometric Asian calls as a function of the diffusion coefficient.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained with the Moreno *et al.* (2018) and Vasicek (1977) models, respectively. In both models, $\lambda = 0$, $r_0 = 0.02$, $k = 0.2$. In the Vasicek (1977) model, $\mu = 0.05$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

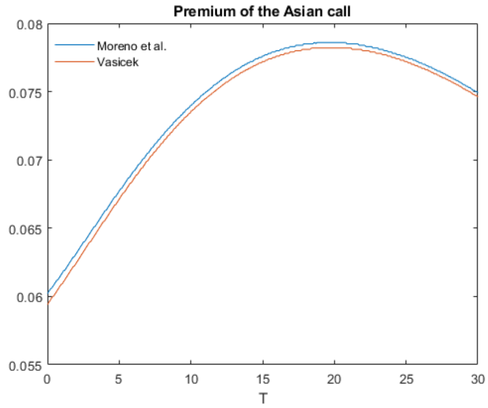

Figure 5: **Premium of geometric Asian calls as a function of the option maturity.** We consider $X = 0.2$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained with the Moreno *et al.* (2018) and Vasicek (1977) models, respectively. In both models, $\lambda = 0$, $r_0 = 0.02$, $k = 0.2$, $\sigma = 0.002$. In the Vasicek (1977) model, $\mu = 0.05$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

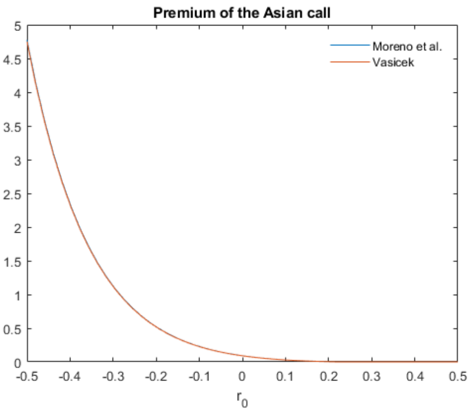

Figure 6: **Premium of geometric Asian calls as a function of the initial instantaneous interest rate.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained with the Moreno *et al.* (2018) and Vasicek (1977) models, respectively. In both models, $\lambda = 0$, $k = 0.2$, $\sigma = 0.002$. In the Vasicek (1977) model, $\mu = 0.05$. In the Moreno *et al.* (2018) model, the remaining parameters are: $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

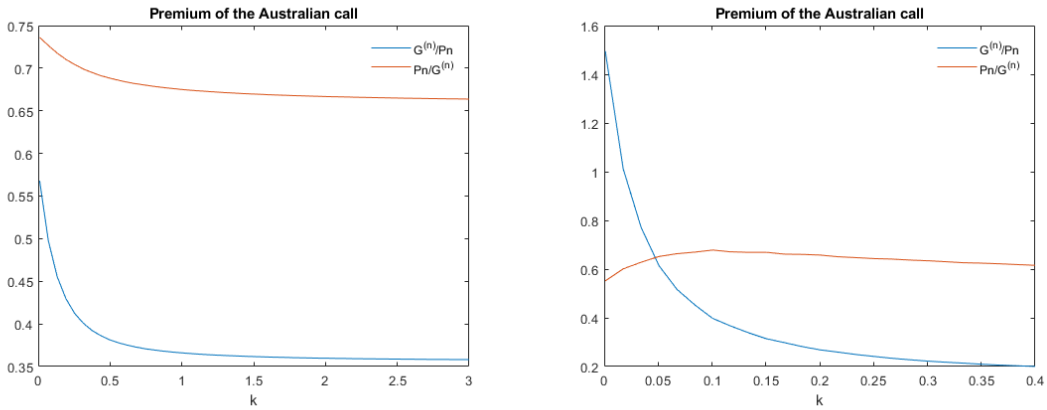

Figure 7: **Premium of geometric Australian calls as a function of the speed of mean-reversion in interest rates.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained in the Moreno *et al.* (2018) model for Australian calls with ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$, respectively. The parameters are $\lambda = 0$, $r_0 = 0.02$, $\sigma = 0.002$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$. The first graph (resp., second) considers $\alpha = 0.05$ (resp., $\alpha = 0.1$) in this model.

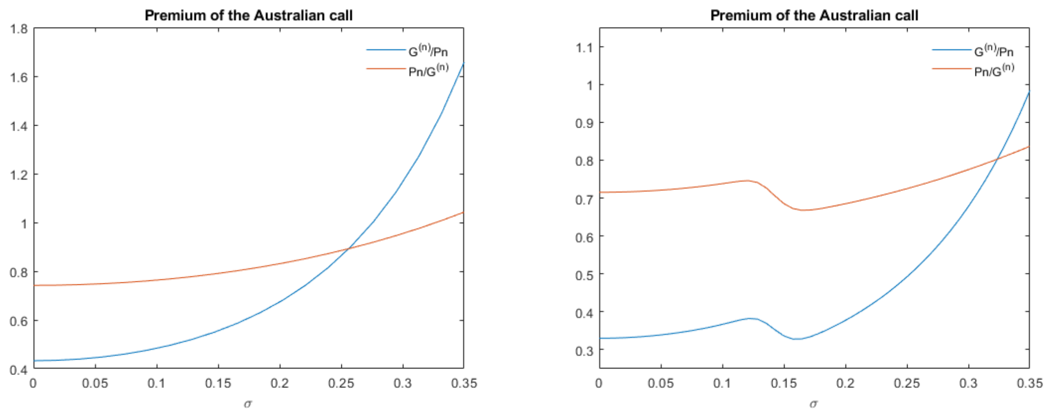

Figure 8: **Premium of geometric Australian calls as a function of the diffusion coefficient.** We consider $X = 0.1$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained in the Moreno *et al.* (2018) model for Australian calls with ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$, respectively. The parameters are $\lambda = 0$, $r_0 = 0.02$, $k = 0.8$, $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$. The first graph i(resp., second) considers $\alpha = 0.05$ (resp., $\alpha = 0.08$) in this model.

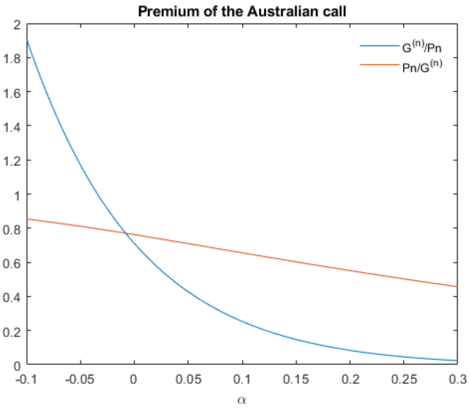

Figure 9: **Premium of geometric Australian calls as a function of the constant term in the level of mean-reversion in interest rates.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained in the Moreno *et al.* (2018) model for Australian calls with ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$, respectively. The parameters are $\lambda = 0$, $r_0 = 0.02$, $k = 0.2$, $\sigma = 0.002$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.

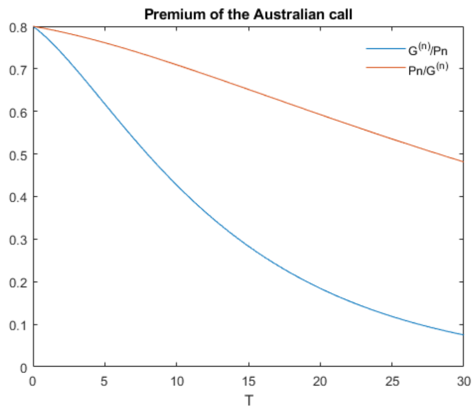

Figure 10: **Premium of geometric Australian calls as a function of the option maturity.** We consider $X = 0.2$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained in the Moreno *et al.* (2018) model for Australian calls with ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$, respectively. The parameters are $\lambda = 0$, $r_0 = 0.02$, $k = 0.2$, $\sigma = 0.002$, $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$

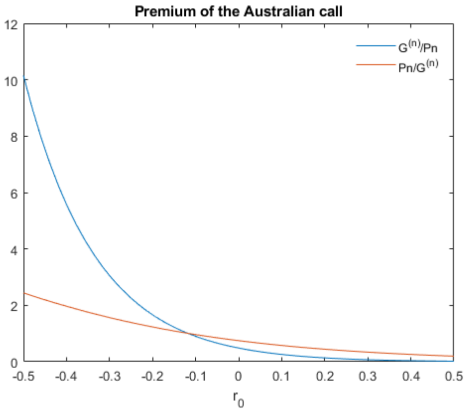

Figure 11: **Premium of geometric Australian calls as a function of the instantaneous interest rate.** We consider $X = 0.2$, $T = 10$, $T_b = 30$, and $n = 100$. Blue and red lines represent the results obtained in the Moreno *et al.* (2018) model for Australian calls with ratios $\frac{G^{(n)}}{P_n}$ and $\frac{P_n}{G^{(n)}}$, respectively. The parameters are $\lambda = 0$, $k = 0.2$, $\sigma = 0.002$, $\alpha = 0.05$, $\omega = 20$, $A_{1,x} = 0.1758$, $A_{1,y} = 0.0402$, $A_{2,x} = -0.3011$, $A_{2,y} = 0.0172$, $A_{3,x} = 0.0498$, $A_{3,y} = -0.1215$, $A_{4,x} = 0.0798$, $A_{4,y} = 0.1618$, $A_{5,x} = 0.0894$, $A_{5,y} = 0.0655$.